Fill Ups of Quadratic Equation and Inequations

Q.1. The coefficient of x^{99} in the polynomial (x – 1) (x – 2) ....(x – 100) is .... (1982 - 2 Marks)

Ans. Sol. Given polynomial :

$(x - 1)(x - 2)(x - 3) \ldots (x - 100)$

$= x^{100} - (1+2+3+\ldots+100)x^{99} + (\ldots)x^{98} + \ldots$

Here coeff. of $x^{99} = -(1+2+3+\ldots+100)$

$= \frac{-100 \times 101}{2} = -5050$

Q.2. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then (p, q) = (.................. , ...). (1982 - 2 Marks)

Ans. Sol. As p and q are real;and one root is $2 + i\sqrt{3}$, other should be $2 - i\sqrt{3}$

Then p = – (sum of roots) = – 4,

q = product of roots = 4 + 3 = 7.

Q.3. If the product of the roots of the equation $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$ is 7, then the roots are real fork = ... (1984 - 2 Marks)

Ans. Sol. The given equation is $x^2 - 3kx + 2e^{2\ln k} - 1 = 0$

Or $x^2 - 3kx + (2k^2 - 1) = 0$

Here product of roots = $2k^2 - 1$

$\therefore 2k^2 - 1 = 7 \Rightarrow k^2 = 4 \Rightarrow k = 2, -2$

Now for real roots we must have $D \geq 0$

$\Rightarrow 9k^2 - 4(2k^2 - 1) \geq 0 \Rightarrow k^2 + 4 \geq 0$

Which is true for all k. Thus k = 2, – 2

But for k = –2, ln k is not define

∴ Rejecting k = –2, we get k = 2

Q.4. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ (a ¹b) have a common root, then the numerical value of a + b is .... (1986 - 2 Marks)

Ans. Sol. ∵ x = 1 reduces both the equations to 1 + a + b = 0

∴ 1 is the common root. for a + b = –1

∴ Numerical value of a + b = 1

Q.5. The solution of equation log7 log5 $(\sqrt{x+5}+\sqrt{x}) = 0$ is .... (1986 - 2 Marks)

Ans. Sol. $\log_7 \log_5 \left(\sqrt{x+5}+\sqrt{x}\right)=0$

$\Rightarrow \log_5\left(\sqrt{x+5}+\sqrt{x}\right)=1$ NOTE THIS STEP

$\Rightarrow \sqrt{x+5}+\sqrt{x}=5 \Rightarrow x+5 = 25+ x-10\sqrt{x}$

$\Rightarrow 2 = \sqrt{x} \Rightarrow x=4$ which satisfies the given equation.

Q.6. If x < 0, y < 0, x + y + $\frac{x}{y} = \frac{1}{2}$ and (x + y) $\frac{x}{y} = -\frac{1}{2}$, then x = ... and y = ... (1990 - 2 Marks)

Ans. Sol. Given x < 0, y < 0

$x+y+\frac{x}{y}=\frac{1}{2}$ and $(x+y).\frac{x}{y}=-\frac{1}{2}$

Let x + y = a and $\frac{x}{y}=b$ .... (1)

∴ We get a + b = $\frac{1}{2}$ and ab = $-\frac{1}{2}$

Solving these two, we get $a+\left(-\frac{1}{2a}\right)=\frac{1}{2}$

$\Rightarrow 2a^2 - a - 1=0 \Rightarrow a = 1, -1/2 \Rightarrow b = -1/2, 1$

$\therefore$ (1) $\Rightarrow x + y = 1$ and $\frac{x}{y} = -\frac{1}{2}$

or $x + y = \frac{-1}{2}$ and $\frac{x}{y} = 1$ But $x, y < 0$

$\therefore x + y < 0 \Rightarrow x + y = \frac{-1}{2}$ and $\frac{x}{y} = 1$

On solving, we get $x = -1/4$ and $y = -1/4$.

Q.7. Let n and k be positive such that $n \geq \frac{k(k+1)}{2}$. The number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1$, $x_2 \geq 2$, $\ldots, x_k \geq k$, all integers, satisfying $x_1 + x_2 + \ldots + x_k = n$, is .... (1996 - 2 Marks)

Ans. Sol. We have $x_1 + x_2 + \ldots\ldots\ldots + x_k = n$ .... (1)

where $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, \ldots\ldots\ldots, x_k \geq k$; all integers

Let $y_1 = x_1 - 1$, $y_2 = x_2 - 2 \ldots\ldots\ldots\ldots\ldots\ldots y_k = x_k - k$

so that $y_1, y_2, \ldots\ldots\ldots, y_k \geq 0$

Substituting the values of $x_1, x_2, \ldots\ldots\ldots\ldots, x_k$ in equation .. (1)

We get $y_1 + y_2 + \ldots\ldots\ldots y_k = n - (1 + 2 + 3K + k)$

$= n - \frac{k(k+1)}{2}$ .... (2)

Now keeping in mind that number of solutions of the equation

$\alpha + 2\beta + 3\gamma + \ldots\ldots q\theta = n$

for $\alpha, \beta, \gamma, \ldots\ldots\ldots\theta \in I$ and each is ≥ 0, is given by coeff of x^n in

$(1 + x + x^2 + \ldots\ldots)(1 + x^2 + x^4 + \ldots\ldots)$

$(1 + x^3 + x^6 + \ldots\ldots)\ldots\ldots(1 + x^q + x^{2q} + \ldots\ldots\ldots)$

We find that no. of solutions of equation (2)

$= \text{coeff of } x^{n-\frac{k(k+1)}{2}} \text{ in } (1+x+x^2+........)^k$

NOTE THIS STEP

$= \text{coeff of } x^{n-\frac{k(k+1)}{2}} \text{ in } (1-x)^{-k}$

$= \text{coeff of } x^{n-\frac{k(k+1)}{2}} \text{ in} (1+{}^kC_1 x+{}^{k+1}C_2 x^2$

$+{}^{k+2}C_3 x^3+.............) = {}^{k+\left(n-\frac{k(k+1)}{2}\right)-1}C_{n-\frac{k(k+1)}{2}}$

$$= \frac{\left[k+\left(n-\frac{k(k+1)}{2}\right)-1\right]!}{\left[n-\frac{k(k+1)}{2}\right]!(k-1)!}.$$

Q. 8. The sum of all the real roots of the equation $|x-2|^2+|x-2|-2=0$ is .... (1997 - 2 Marks)

Ans. Sol. $|x-2|^2+|x-2|-2=0$

Case 1. $x \geq 2$

$\Rightarrow (x-2)^2+(x-2)-2=0$

$\Rightarrow x^2-3x=0 \Rightarrow x(x-3)=0$

$\Rightarrow x = 0, 3$ (0 is rejected as $x \geq 2$)

$\Rightarrow x = 3$ ....(1)

Case 2. $x < 2$

$\{-(x-2)\} - (x-2) - 2 = 0$

$\Rightarrow x^2+4-4x-x=0 \Rightarrow (x-1)(x-4)=0$

$\Rightarrow x = 1, 4$ (4 is rejected as $x < 2$)

$\Rightarrow x = 1$ ....(2)

Therefore, the sum of the roots is 3 + 1= 4.

True False of Quadratic Equation and Inequations

Q.1. For every integer n > 1, the inequality $(n!)^{1/n} < \frac{n+1}{2}$ holds. **(1981 – 2 Marks)**

Ans. T

Sol. Consider n numbers, namely 1, 2, 3, 4, ...n.

KEY CONCEPT : Now using A.M. > G.M. for distinct numbers, we get

$$\frac{1+2+3+4....+n}{n} > (1.2.3.4..n)^{1/n}$$

$$\Rightarrow \frac{n(n+1)}{2n} > (n!)^{1/n} \Rightarrow (n!)^{1/n} < \frac{n+1}{2} \therefore True$$

Q.2. The equation $2x^2 + 3x + 1 = 0$ has an irrational root. (1983 - 1 Mark)

Ans. F

Sol. $2x^2 + 3x + 1 = 0 \Rightarrow x = -1, -1/2$ both are rational

∴ Statement is FALSE.

Q.3. If a < b < c < d, then the r oots of the equation

(x – a) (x – c) + 2(x – b) (x – d) = 0 are real and distinct. (1984 - 1 Mark)

Ans. T

Sol. f(x) = (x – a) (x – c) + 2 (x – b) (x – d).

f (a) = + ve; f (b) = – ve ; f(c) = – ve ;

f (d) = + ve

∴ There exists two real and distinct roots one in the interval (a, b) and other in (c, d). Hence, (True).

Q.4. If n_1, n_2, ...n_p are p positive integers, whose sum is an even number, then the number of odd integers among them is odd. (1985 - 1 Mark)

Ans. F

Sol. Consider $N = n_1 + n_2 + n_3 + ... + n_p$, where N is an even number.

Let k numbers among these p numbers be odd, then p – k are even numbers.

Now sum of (p – k) even numbers is even and for N to be an even number, sum of k odd numbers must be even which is possible only when k is even.

∴ The given statement is false.

Q.5. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)Q(x)=0$ has at least two real roots. (1985 - 1 Mark)

Ans. T

Sol. $P(x).Q(x) = (ax^2 + bx + c)(-ax^2 + bx + c)$

$\Rightarrow D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$

clearly, $D1 + D_2 = 2b^2 \geq 0$

∴ atleast one of D_1 and D_2 is (+ ve). Hence, atleast two real roots.

Thus, (True)

Q.6. If x and y are positive real numbers and m, n are any positive integers, then $\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} > \frac{1}{4}$ (1989 - 1 Mark)

Ans. F

Sol. As x and y are positive real numbers and m and n are positive integers

$\therefore \quad \frac{1+x^{2n}}{2} \geq (1 \times x^{2n})^{1/2}$ and $\frac{1+y^{2m}}{2} \geq (1 \times y^{2m})^{1/2}$

{For two +ve numbers A.M. ≥ G.M.}

$$\Rightarrow \left(\frac{1+x^{2n}}{2}\right) \geq x^n \ldots.(1)$$

and $\left(\frac{1+y^{2m}}{2}\right) \geq y^m \ldots.(2)$

Multiplying (1) and (2), we get

$$\frac{(1+x^{2n})(1+y^{2m})}{4} \geq x^n y^m \Rightarrow \frac{1}{4} \geq \frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}$$

Hence the statement is false.

Subjective questions of Quadratic Equation and Inequations

Q.1. Solve for x : $4^x - 3^{x-\frac{1}{2}} = 3^{x+\frac{1}{2}} - 2^{2x-1}$ **(1978)**

Ans.

Sol. $4^x - 3^{x-1/2} = 3^{x+1/2} - \frac{(2^2)^x}{2}$

$\Rightarrow 4^x - \frac{3^x}{\sqrt{3}} = 3^x\sqrt{3} - \frac{4^x}{2}$

$\Rightarrow \frac{3}{2}\cdot 4^x = 3^x\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right) \Rightarrow \frac{3}{2}\cdot 4^x = 3^x \frac{4}{\sqrt{3}}$

$\Rightarrow \frac{4^{x-1}}{4^{1/2}} = \frac{3^{x-1}}{\sqrt{3}} \Rightarrow 4^{x-3/2} = 3^{x-3/2}$

$\Rightarrow \left(\frac{4}{3}\right)^{x-3/2} = 1 \Rightarrow x - \frac{3}{2} = 0 \Rightarrow x = 3/2$

Q.2.

If (m , n) = $\frac{(1-x^m)(1-x^{m-1})..........(1-x^{m-n+1})}{(1-x)(1-x^2)............(1-x^n)}$ **(1978)**

where m and n are positive integers ($n \le m$), show that (m, n + 1) = (m – 1, n + 1) + x^{m-n-1} (m – 1, n).

Ans.

Sol. RHS = (m – 1, n + 1) + x^{m-n-1} (m - 1,n)

$= \frac{(1-x^{m-1})(1-x^{m-2})...(1-x^{m-n-1})}{(1-x)(1-x^2)...(1-x^{n+1})}$

$+x^{m-n-1}\left[\frac{(1-x^{m-1})(1-x^{m-2})...(1-x^{m-n})}{(1-x)(1-x^2)...(1-x^n)}\right]$

$= \frac{(1-x^{m-1})(1-x^{m-2})...(1-x^{m-n})}{(1-x)(1-x^2)...(1-x^n)}$

$$\left[\frac{1-x^{m-n-1}}{1-x^{n+1}}+x^{m-n-1}\right]$$

$$\left[\frac{1-x^{m-n-1}+x^{m-n-1}-x^{m}}{1-x^{n+1}}\right]$$

$$=\frac{(1-x^{m})(1-x^{m-1})...(1-x^{m-n})}{(1-x)(1-x^{2})...(1-x^{n})(1-x^{n+1})}$$

= (m, n +1) = L.H.S. Hence Proved

Q.3. Solve for x : $\sqrt{x+1}-\sqrt{x-1}=1$. (1978)

Ans.

Sol. $\sqrt{x+1}=1+\sqrt{x-1}$

Squaring both sides, we get

x +1 = 1+ x -1+ $2\sqrt{x-1} \Rightarrow 1=2\sqrt{x-1}$.

$\Rightarrow 1 = 4(x-1)$

$\Rightarrow x = 5/4$

Q.4. Solve the following equation for x : (1978)

$$2\log_x a+\log_{ax} a+3\log_{a^2x} a=0,\ a>0$$

Ans.

Sol. Given a > 0, so we have to consider two cases :

$a \neq 1$and $a = 1$. Also it is clear that $x > 0$ and $x \neq 1$, $ax \neq 1$, $a^2x \neq 1$.

Case I : If $a > 0, \neq 1$

then given equation can be simplified as

$$\frac{2}{\log_a x}+\frac{1}{1+\log_a x}+\frac{3}{2+\log_a x}=0$$

Putting $\log_a x = y$, we get

$2(1 + y)(2 + y) + y(2 + y) + 3y(1 + y) = 0$

$\Rightarrow 6y^2 + 11y + 4 = 0 \Rightarrow y = -4/3$ and $-1/2$

$\Rightarrow \log_a x = -4/3$ and $\log_a x = -1/2$

$\Rightarrow x = a^{-4/3}$ and $x = a^{-1/2}$

Case II : If a = 1 then equation becomes

$2\log_x 1 + \log_x 1 + 3\log_x 1 = 6\log_x 1 = 0$

which is true $\forall\ x > 0, \neq 1$

Hence solution is if $a = 1$, $x > 0, \neq 1$

if $a > 0, \neq 1$; $x = a^{-1/2}, a^{-4/3}$

Q.5. Show that the square of $\frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$ is a rational number. (1978)

Ans. Sol.

Let $x = \frac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{50+38+5\sqrt{3}-10\sqrt{76+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{75+1+10\sqrt{3}}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3})^2+(1)^2+2\times 5\sqrt{3}\times 1}}$$

$$\Rightarrow x^2 = \frac{26-15\sqrt{3}}{88+5\sqrt{3}-10\sqrt{(5\sqrt{3}+1)^2}}$$

$= \frac{26-15\sqrt{3}}{3(26-15\sqrt{3})} = \frac{1}{3}$, which is a rational number..

Q.6. Sketch the solution set of the followin g system of inequalities: $x^2 + y^2 - 2x \geq 0$; $3x - y - 12 \leq 0$; $y - x \leq 0$; $y \geq 0$. (1978)

Ans.
Sol. $x^2 + y^2 - 2x \geq 0 \Rightarrow x^2 - 2x + 1 + y^2 \geq 1$

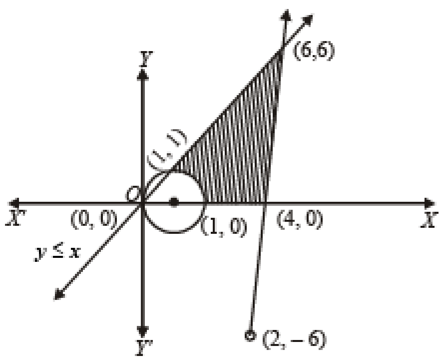
$\Rightarrow (x - 1)^2 + y^2 \geq 1$ which represents the boundary and exterior region of the circle with centre at (1,0) and radius as 1.

For $3x - y \leq 12$, the corresponding equation is $3x - y = 12$; any two points on it can be taken as (4, 0), (2, – 6). Also putting (0, 0) in given inequation, we get $0 \leq 12$ which is true.

$\therefore$ given inequation represents that half plane region of line $3x - y = 12$ which contains origin.

For $y \leq x$, the corresponding equation $y = x$ has any two points on it as (0, 0) and (1, 1). Also putting (2, 1) in the given i nequation, we get $1 \leq 2$ wh ich is true, so $y \leq x$ represents that half plane which contains the points (2, 1). $y \geq 0$ represents upper half cartesian plane.

Combining all we find the solution set as the shaded region in the graph.



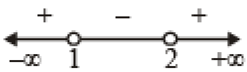
Q.7. Find all integers x for which (1978)

$(5x - 1) < (x + 1)^2 < (7x - 3)$.

Ans.
Sol. There are two parts of this question $(5x - 1) < (x + 1)^2$ and $(x + 1)^2 < (7x - 3)$

Taking first part $(5x - 1) < (x + 1)^2 \Rightarrow 5x - 1 < x^2 + 2x + 1$

$\Rightarrow x^2 - 3x + 2 > 0 \Rightarrow (x-1)(x-2)>0$



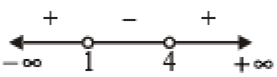
(using wavy method)

$\Rightarrow x < 1$ or $x > 2$ ....(1)

Taking second part

$(x+1)^2 < (7x-3) \Rightarrow x^2 - 5x + 4 < 0$

$\Rightarrow (x-1)(x-4) < 0$



(using wavy method)

$\Rightarrow 1 < x < 4$ ....(2)

Combining (1) and (2) [taking common solution], we get $2 < x < 4$ but x is an integer therefore $x = 3$.

Q.8. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s.

Deduce the condition that the equations have a common root.
(1979)

Sol. ∵ α, β are the roots of $x^2 + px + q = 0$

$\therefore\ \alpha + \beta = -p, \quad \alpha\beta = q$

∵ γ,δ are the roots of $x^2 + rx + s = 0$

$\therefore \gamma + \delta = -r,\ \gamma\delta = s$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta$

$= [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta][\beta^2 - (\gamma + \delta)\beta + \gamma\delta]$

$= [\alpha^2 + r\alpha + s][\beta^2 + r\beta + s]$

[∵ α , β are roots of $x^2 + px + q = 0$

$\therefore \alpha^2 + p\alpha + q = 0$ and $\beta^2 + p\beta + q = 0$]

$= [(r - p)\alpha + (s - q)][(r - p)\beta + (s - q)]$

$= (r - p)^2 \alpha\beta + (r - p)(s - q)(\alpha + \beta) + (s - q)^2$

$= q(r - p)^2 - p(r - p)(s - q) + (s - q)^2$

Now if the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root say α, then $\alpha^2 + p\alpha + q = 0$ and $\alpha^2 + r\alpha + s = 0$

$$\Rightarrow \frac{\alpha^2}{ps - qr} = \frac{\alpha}{q - s} = \frac{1}{r - p}$$

$$\Rightarrow \alpha^2 = \frac{ps - qr}{r - p} \text{ and } \alpha = \frac{q - s}{r - p}$$

$\Rightarrow (q - s)2 = (r - p)(ps - qr)$ which is the required condition.

Q.9. Given $n^4 < 10^n$ for a fixed positive integer $n \geq 2$, prove that $(n + 1)^4 < 10^{n+1}$. (1980)

Ans.

Sol. Given that $n^4 < 10^n$ for a fixed +ve integer $n \geq 2$.

To prove that $(n + 1)^4 < 10^{n+1}$

Proof : Since $n^4 < 10^n \Rightarrow 10n^4 < 10^{n+1}$ ....(1)

So it is sufficient to prove that $(n + 1)^4 < 10n^4$

$$\text{Now } \left(\frac{n+1}{n}\right)^4 = \left(1 + \frac{1}{n}\right)^4 \leq \left(1 + \frac{1}{2}\right)^4 \quad [\because n \geq 2]$$

$$= \frac{81}{16} < 10$$

$\Rightarrow (n + 1)^4 < 10n^4$ .... (2)

From (1) and (2), $(n + 1)^4 < 10^{n+1}$

Q.10. Let $y = \sqrt{\frac{(x+1)\,(x-3)}{(x-2)}}$ **(1980)**

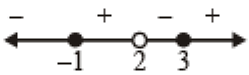
Find all the real values of x for which y takes real values.

Ans. Sol.

$$y = \sqrt{\frac{(x+1)(x-3)}{(x-2)}}$$

y will take all real values if $\frac{(x+1)(x-3)}{(x-2)} \geq 0$

By wavy method



$x \in [-1, 2) \cup [3, \infty)$

[2 is not included as it makes denominator zero, and hence y an undefined number.]

Q.11. For what values of m, does the system of equations

3x + my = m
2x – 5y = 20

has solution satisfying the conditions x > 0, y > 0. (1980)

Ans. Sol. The given equations are 3x + my – m = 0 and 2x – 5y – 20 = 0 Solving these equations by cross product method, we get

$$\frac{x}{-20m-5m} = \frac{y}{-2m+60} = \frac{1}{-15-2m}$$ NOTE THIS STEP

$$\Rightarrow \quad x = \frac{25m}{2m+15}, y = \frac{2m-60}{2m+15}$$

For $x > 0 \Rightarrow \frac{25m}{2m+15} > 0$ ....(1)

$$\Rightarrow m < -\frac{15}{2} \text{ or } m > 0$$

For $y > 0 \Rightarrow \frac{2(m-30)}{2m+15} > 0$ ....(2)

$\Rightarrow m < -\frac{15}{2}$ or $m > 30$

Combining (1) and (2), we get the common values of m as follows :

$m < -\frac{15}{2}$ or $m > 30$ $\therefore$ $m \in \left(-\infty, \frac{-15}{2}\right) \cup (30, \infty)$

Q.12. Find the solution set of the system (1980)

x + 2y + z = 1;
2x – 3y – w = 2;

$x \geq 0; y \geq 0; z \geq 0; w \geq 0.$

Ans. Sol. The given system is

x + 2y + z = 1 ....(1)
2x - 3y - ω =2 ....(2)
where x, y, z, ω ≥ 0

Multiplying eqn. (1) by 2 and subtracting from (2), we get

$7y + 2z + \omega = 0 \Rightarrow \omega = -(7y + 2z)$

Now if y, z > 0, ω < 0 (not possible)

If y = 0, z = 0 then x = 1 and ω = 0 .

∴ The only solution is x = 1, y = 0, z = 0, ω = 0 .

Q.13. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. (1982 - 2 Marks)

Ans.

Sol. $e^{\sin x} - e^{-\sin x} - 4 = 0$

Let $e^{\sin x} = y$ then $e^{-\sin x} = 1/y$

$\therefore$ Equation becomes, $y - \frac{1}{y} - 4 = 0$

$\Rightarrow y^2 - 4y - 1 = 0 \Rightarrow y = 2+\sqrt{5}, 2-\sqrt{5}$

But y is real +ve number,

$\therefore \quad y \neq 2-\sqrt{5} \Rightarrow y = 2+\sqrt{5}$

$\Rightarrow e^{\sin x} = 2+\sqrt{5} \Rightarrow \sin x = \log_e (2+\sqrt{5})$

But $2+\sqrt{5} > e \Rightarrow \log_e(2+\sqrt{5}) > \log_e e$

$\Rightarrow \log_e(2+\sqrt{5}) > 1$ Hence, $\sin x > 1$

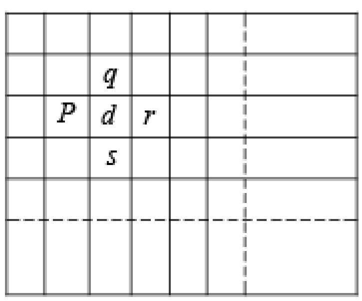
Which is not possible.

$\therefore$ Given equation has no real solution.

Q.14. mn squares of euqal size are arranged to from a rectangle of dimension m by n, where m and n are natural numbers.Two squares will be called 'neighbours' if they have exactly one common side. A natural number is written in each square such that the number written in any square is the arithmetic mean of the numbers written in its neighbouring squares.

Show that this is possible only if all the numbers used are equal. (1982 - 5 Marks)

Ans. Sol. For any square there can be at most 4, neighbouring squares.



Let for a square having largest number d, p, q, r, s be written then

According to the question, p + q + r + s = 4d

$\Rightarrow (d - p) + (d - q) + (d - r) + (d - s) = 0$

Sum of four +ve numbers can be zero only if these are zero individually

$\therefore$ d – p = 0 = d – q = d – r = d – s

$\Rightarrow$ p = q = r = s = d

$\Rightarrow$ all the numbers written are same.
Hence Proved.

Q.15. If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the n-th power of the other, then show that

$$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$$ **(1983 - 2 Marks)**

Ans. Sol.

Let α, β be the roots of eq. $ax^2 + bx + c = 0$

According to the question, $\beta = a^n$

Also α + β = – b/a ; αβ = c/a

$$\alpha\beta = \frac{c}{a} \Rightarrow \alpha.\alpha^n = \frac{c}{a} \Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$$

then α + β = – b/a $\Rightarrow$ $\alpha + \alpha^n = \frac{-b}{a}$

or $\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + \left(\frac{c}{a}\right)^{\frac{n}{n+1}} = \frac{-b}{a}$

$$\Rightarrow a.\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + a.\left(\frac{c}{a}\right)^{\frac{n}{n+1}} + b = 0$$

$$\Rightarrow a^{\frac{n}{n+1}} c^{\frac{1}{n+1}} + a^{\frac{1}{n+1}} c^{\frac{n}{n+1}} + b = 0$$

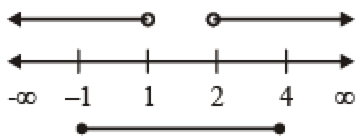
$$\Rightarrow (a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} + b = 0$$

Hence Proved.

Q.16. Find all real values of x which satisfy x^2 - 3x + 2 > 0 and x^2 - 2 x - 4 ≤ 0 (1983 - 2 Marks)

Ans. Sol. $x^2 - 3x + 2 > 0,\ x^2 - 3x - 4 \le 0$

$\Rightarrow (x-1)(x-2) > 0$ and $(x-4)(x+1) < 0$



$x \in (-\infty, 1) \cup (2, \infty)$ and $x \in [-1, 4]$

$\therefore$ Common solution is $[-1, 1) \cup (2, 4]$

Q.17. Solve for x; $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3}$ **= 10 (1985 - 5 Marks)**

Ans. Sol. The given equation is

$(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$ ....(1)

Let $(5+2\sqrt{6})^{x^2-3} = y$....(2)

then $(5-2\sqrt{6})^{x^2-3} = \left(\frac{(5-2\sqrt{6})(5+2\sqrt{6})}{5+2\sqrt{6}}\right)^{x^2-3}$

$= \left(\frac{25-24}{5+2\sqrt{6}}\right)^{x^2-3} = \left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3} = \frac{1}{y}$ (Using (2))

$\therefore$ The given equation (1) becomes $y + \frac{1}{y} = 10$

$\Rightarrow y^2 - 10y + 1 = 0 \Rightarrow \frac{10 \pm \sqrt{100-4}}{2} = \frac{10 \pm 4\sqrt{6}}{2}$

$\Rightarrow y = 5 + 2\sqrt{6}$ or $5 - 2\sqrt{6}$

Consider, $y = 5 + 2\sqrt{6}$

$\Rightarrow (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})$

$\Rightarrow x^2 - 3 = 1 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

Again consider

$y = 5 - 2\sqrt{6} = \frac{1}{5+2\sqrt{6}} = (5+2\sqrt{6})^{-1}$

$\Rightarrow (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^{-1} \Rightarrow x^2 - 3 = -1$

$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Hence the solutions are 2, – 2, $\sqrt{2}$, -$\sqrt{2}$

Q.18. For a ≤ 0, determine all r eal roots of the equation $x^2 - 2a \mid x - a \mid - 3a^2 = 0$ (1986 - 5 Marks)
Ans.

Sol. The given equation is, $x^2 - 2a \mid x - a \mid - 3a^2 = 0$

Here two cases are possible.

Case I : $x - a > 0$ then $\mid x - a \mid = x - a$

$\therefore$ Eq. becomes $x^2 - 2a(x - a) - 3a^2 = 0$

or $x^2 - 2ax - a^2 = 0 \Rightarrow x = \frac{2a \pm \sqrt{4a^2 + 4a^2}}{2}$

$\Rightarrow x = a \pm a\sqrt{2}$

Case II : $x - a < 0$ then $\mid x - a \mid = -(x - a)$

$\therefore$ Eq. becomes

$x^2 + 2a(x - a) - 3a^2 = 0$

or $x^2 + 2ax - 5a^2 = 0 \Rightarrow x = \frac{-2a \pm \sqrt{4a^2 + 20a^2}}{2}$

$\Rightarrow x = \frac{-2a \pm 2a\sqrt{6}}{2} \Rightarrow x = -a \pm a\sqrt{6}$

Thus the solution set is $\{a \pm a\sqrt{2}, -a \pm a\sqrt{6}\}$

Q.19. Find the set of all x for which $\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$ **(1987 - 3 Marks)**

Ans. Sol. We are given $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$

$$\Rightarrow \frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$$

$$\Rightarrow \frac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$$

$$\Rightarrow \frac{-3x-2}{(2x+1)(x+1)(x+2)} > 0 \Rightarrow \frac{(3x+2)}{(x+1)(x+2)(2x+1)} < 0$$

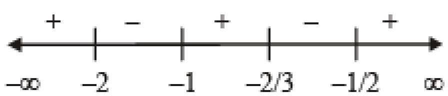
$$\Rightarrow \frac{(3x+2)(x+1)(x+2)(2x+1)}{(x+1)^2(x+2)^2(2x+1)^2} < 0$$

$\Rightarrow (3X+2)(X+1)(X+2)(2X+1) < 0$ ....(1)

NOTE THIS STEP

: Critical pts are x = – 2/3, –1, – 2, – 1/2

On number line



Clearly Inequality (1) holds for,

$x \in (-2, -1) \cup (-2/3, -1/2)$

[as $x \neq -2, -1, -2/3, -1/2$]

Q.20. Solve | x^2 + 4 x + 3 | +2x + 5=0 (1988 - 5 Marks)

Ans. Sol. The Given equation is, | x^2 + 4x + 3 | + 2x + 5 = 0 Now there can be two cases.

Case I : $x^2 + 4x + 3 \geq 0 \Rightarrow (x+1)(x+3) \geq 0 \Rightarrow x \in (-\infty, -3] \cup [-1, \infty)$ ....(i)

Then given equation becomes,

$\Rightarrow x^2 + 6x + 8 = 0$

$\Rightarrow$ (x + 4)(x + 2)= 0 $\Rightarrow$ x = – 4, – 2

But x = – 2 does not satisfy (i), hence rejected

$\therefore$ x = – 4 is the sol.

Case II : $x^2 + 4x + 3 < 0$

$\Rightarrow (x + 1)(x + 3) < 0$

$\Rightarrow x \in (-3, -1)$ ....(ii)

Then given equation becomes, $-(x^2 + 4x + 3) + 2x + 5 = 0$

$\Rightarrow -x^2 - 2x + 2 = 0 \Rightarrow x^2 + 2x - 2 = 0$

$x = \frac{-2 \pm \sqrt{4+8}}{2} \Rightarrow \quad x = -1+\sqrt{3}, -1-\sqrt{3}$

Out of which $x = -1 - \sqrt{3}$ is sol.

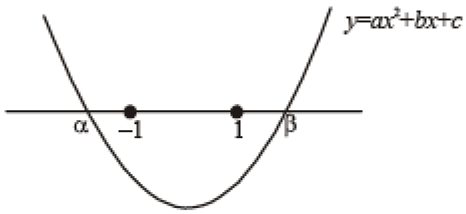
Combining the two cases we get the solutions of given equation as $x = -4, -1-\sqrt{3}$

Q.21. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β, where α < -1 and β > 1 , then show that

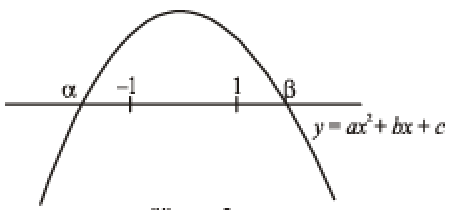
$1+\frac{c}{a}+\left|\frac{b}{a}\right|<0$. **(1995 - 5 Marks)**

Ans. Sol. Given that for a, b, c ∈ R, $ax^2 + bx + c = 0$ has two real roots a and b, where a < – 1 and b > 1. There may be two cases depending upon value of a, as shown below.

In each of cases (i) and (ii) af (–1) < 0 and af (1) < 0



(i) $a > 0$



(ii) $a < 0$

$\Rightarrow$ a (a – b + c) < 0 and a (a + b + c) < 0 Dividing by a^2 (> 0), we get

$1-\frac{b}{a}+\frac{c}{a}<0$ ....(1)

and $1+\frac{b}{a}+\frac{c}{a}<0$ ....(2)

Combining (1) and (2) we get

$1+\left|\frac{b}{a}\right|+\frac{c}{a}<0$ or $1+\frac{c}{a}+\left|\frac{b}{a}\right|<0$ Hence Proved.

Q.22. Let S be a square of unit area. Consider any quadrilateral which has one vertex on each side of S. If a, b, c, and d denote the lengths of the sides of the quadrilateral, prove that $2 \le a^2+b^2+c^2+d^2 \le 4$. (1997 - 5 Marks)

Ans.

Sol. $a^2 = p^2 + s^2$, $b^2 = (1- p)^2 + q^2$

$c^2 = (1- q)^2 + (1 - r)^2$,

$d^2 = r^2 + (1- s)^2$

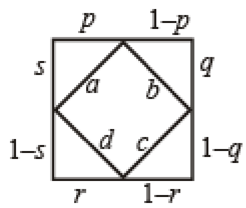
$\therefore$ a2 + $b^2 + c^2 + d^2 = \{p^2 + (1- p)^2\}+\{q^2 - (1- q)^2\}$

$+ \{ r^2 + (1- r)^2 \} +\{s^2 + (1- s)^2 \}$

where p, q, r, s all vary in the interval [0, 1].

Now consider the function

$y^2 = x^2 + (1-x)^2,\ 0 \le x \le 1,$



$$2y\frac{dy}{dx} = 2x - 2(1-x) = 0$$

$$\Rightarrow\ x = \frac{1}{2} \text{ which } \frac{d^2y}{dx^2} = 4 \text{ i.e. +ive}$$

Hence y is minimum at $x = \frac{1}{2}$ and its minimum

value is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Clearly value is maximum at the end pts which is 1.

∴ Minimum value of $a^2 + b^2 + c^2 + d^2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$ and maximum value is 1 + 1+ 1 + 1= 4. Hence proved.

Q.23. If α, β are the roots of $ax^2 + bx + c = 0$, ($a \neq 0$) and α + δ, β + δ are the roots of $Ax^2 + Bx + C = 0$, ($A \neq 0$) for some constant δ, then prove that $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$ (2000 - 4 Marks)

Ans. Sol. We know that, $(\alpha-\beta)^2 = [(\alpha+\delta) - (\beta+\delta)]^2$

$\Rightarrow (a+\beta)^2 - 4\alpha\beta = (\alpha+\delta+\beta+\delta)^2 - 4(\alpha+\delta)(\beta+\delta)$

$$\Rightarrow \frac{b^2}{a^2} - \frac{4c}{a} = \frac{B^2}{A^2} - \frac{4C}{A} \Rightarrow \frac{4ac - b^2}{a^2} = \frac{4AC - B^2}{A^2}$$

[Here $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$,

$(\alpha+\delta)(\beta+\delta)$

$= -\frac{B}{A}$ and $(\alpha+\delta)(\beta+\delta) = \frac{C}{A}$]

Hence proved.

Q.24. Let a, b, c be real numbers with a ≠ 0 and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a3x^2 + abcx + c^3 = 0$ in terms of α, β. (2001 - 4 Marks)

Ans. Sol. Divide the equation by α^3, we get

$$x^2 + \frac{b}{a}.\frac{c}{a}x + \left(\frac{c}{a}\right)^3 = 0$$

$\Rightarrow x^2 - (\alpha + \beta).(\alpha\beta)x + (\alpha\beta)^3 = 0$

$\Rightarrow x^2 - \alpha^2\beta x - \alpha\beta^2 x + (\alpha\beta)^3 = 0$

$\Rightarrow x(x - \alpha^2\beta) - \alpha\beta^2(x - \alpha^2\beta) = 0$

$\Rightarrow (x - \alpha^2\beta)(x - \alpha\beta^2) = 0$

$\Rightarrow x = \alpha^2\beta, \alpha\beta^2$

which is the required answer.

Q.25. If $x^2 + (a - b)x + (1 - a - b) = 0$ where a, b ∈ R then find the values of a for which equation has unequal real roots for all values of b. (2003 - 4 Marks)

Ans. Sol. The given equation is, $x^2 + (a - b)x + (1 - a - b) = 0$, a, b ∈ R

For this eqn to have unequal real roots ∀ b D > 0

$\Rightarrow (a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow b^2 + b(4 - 2a) + a^2 + 4a - 4 > 0$

Which is a quadratic expression in b, and it will be true ∀ b∈R if discriminant of above eqn less than zero.

i.e., $(4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow (2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow 4 - 4a + a^2 - a^2 - 4a + a < 0$

$\Rightarrow -8a + 8 < 0$

$\Rightarrow a > 1$

Q.26. If a, b, c are positive real numbers. Then prove that $(a + 1)^7 (b + 1)^7 (c + 1)^7 > 7^7 a^4 b^4 c^4$ (2004 - 4 Marks)

Ans. Sol. Given that a, b, c are positive real numbers. To prove that $(a + 1)^7 (b + 1)^7 (c + 1)^7 > 7^7 a^4 b^4 c^4$

Consider L.H.S. $= (1+ a)^7 . (1+ b)^7 . (1+ c)^7$

$= [(1 + a)(1+ b)(1+ c)]^7 [1 + a + b + c + ab + bc + ca + abc]^7$

> [a + b + c + ab + bc + ca + abc]7 ....(1)

Now we know that AM ≥ GM using it for +ve no's a, b, c, ab, bc, ca and abc, we get

$$\frac{a+b+c+ab+bc+ca+abc}{7} \geq (a^4b^4c^4)^{1/7}$$

$\Rightarrow (a + b + c + ab + bc + ca + abc)^7 \geq 7^7 (a^4b^4c^4)a$

From (1) and (2),

we get $[(1 + a)(1 + b)(1 + c)]^7 > 7^7 a^4 b^4 c^4$

Hence Proved.

Q.27. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d then the value of a + b + c + d, when a ≠ b ≠ c ≠ d, is. (2006 - 6M)

Ans. Sol. Roots of $x^2 - 10cx - 11d = 0$ are a and b ⇒ $a + b = 10c$ and $ab = -11d$
Similarly c and d are the roots of $x^2 - 10ax - 11b = 0$

⇒ $c + d = 10a$ and $cd = -11b$

⇒ $a + b + c + d = 10(a + c)$ and $abcd = 121\ bd$

⇒ $b + d = 9(a + c)$ and $ac = 121$

Also we have $a^2 - 10ac - 11d = 0$ and $c^2 - 10ac - 11b = 0$

⇒ $a^2 + c^2 - 20ac - 11(b + d) = 0$

⇒ $(a + c)^2 - 22 \times 121 - 99(a + c) = 0$

⇒ $a + c = 121$ or -22 For $a + c = -22$,

we get $a = c$

∴ rejecting this value we have $a + c = 121$

∴ $a + b + c + d = 10(a + c) = 1210$

Integer Type ques of Quadratic Equation and Inequations

Q.1. Let (x, y, z) be points with integer coordinates satisfying the system of homogeneous equations :

$3x - y - z = 0$
$-3x + z = 0$
$-3x + 2y + z = 0$

Then the number of such points for which $x^2 + y^2 + z^2 \leq 100$ is (2009)

Ans. (7)

Sol. The given system of equations is

$3x - y - z = 0$

$-3x + z = 0$

$-3x + 2y + z = 0$

Let x = p

where p is an integer, then y = 0 and z = 3p

But $x^2 + y^2 + z^2 \leq 100$

$\Rightarrow p^2 + 9p^2 \leq 100$

$\Rightarrow p^2 \leq 10 \Rightarrow p = 0, \pm 1, \pm 2 \pm 3$ i.e. p can take 7 different values.

∴ Number of points (x, y, z) are 7.

Q.2. The smallest value of k, for which both the roots of the equation $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is (2009)

Ans. (2)

Sol. The given equation is $x^2 - 8kx + 16(k^2 - k + 1) = 0$

∵ Both the roots are real and distinct

∴ $D > 0$

⇒ $(8k)2 - 4 \times 16(k^2 - k + 1) > 0$

⇒ $k > 1$...(i)

∵ Both the roots are greater than or equal to 4

∴ $\alpha + \beta > 8$ and $f(4) \geq 0 \Rightarrow k > 1$ ...(ii)

and $16 - 32k + 16(k^2 - k + 1) \geq 0$

⇒ $k^2 - 3k + 2 \geq 0$

⇒ $(k - 1)(k - 2) \geq 0 \Rightarrow k \in (-\infty, 1] \cup [2, \infty)$ ...(iii)

Combining (i), (ii) and (iii),

we get $k \geq 2$ or the smallest value of $k = 2$.

Q.3. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} where $a > 0$ is (2011)

Ans. (8)

Sol. ∵ $a > 0$, ∴ $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8, a^{10} > 0$

Using AM > GM for positive real numbers we get

$$\frac{\frac{1}{a^5}+\frac{1}{a^4}+\frac{1}{a^3}+\frac{1}{a^3}+\frac{1}{a^3}+1+a^8+a^{10}}{8} \geq \left(\frac{1}{a^5}\cdot\frac{1}{a^4}\cdot\frac{1}{a^3}\cdot\frac{1}{a^3}\cdot\frac{1}{a^3}.1.a^8.a^{10}\right)^{\frac{1}{8}}$$

$$\Rightarrow \frac{1}{a^5}+\frac{1}{a^4}+\frac{3}{a^3}+1+a^8+a^{10} \geq 8(1)^{\frac{1}{8}}$$

Q.4. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is (2011)

Ans. Sol. (2) We have $x^4 - 4x^3 + 12x^2 + x - 1 = 0$

$\Rightarrow x^4 - 4x^3 + 6x^2 - 4x + 1 + 6x^2 + 5x - 2 = 0$

$\Rightarrow (x - 1)^4 + 6x^2 + 5x - 2 = 0$

$\Rightarrow (x - 1)^4 = - 6x^2 - 5x + 2$

To solve the above polynomial, it is equivalent to find the intersection points of the curves $y = (x - 1)^4$ and $y = - 6x^2 - 5x + 2$ or $y = (x - 1)^4$ and $\left(x+\frac{5}{12}\right)^2 = -\frac{1}{6}\left(y-\frac{73}{24}\right)$

The graph of above two curves as follows.

Clearly they have two points of intersection.

Hence the given polynomial has two real roots